$e$–filters in Stone Almost Distributive Lattices

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Abstract: The concept of $e$–filters is introduced in a Stone Almost Distributive lattice and element wise characterization is developed for $e$–filters. Several properties are derived on $e$–filters with the help of maximal filters. It is also proved that the set of all $e$–filters forms a complete distributive lattice.

Keywords: Almost Distributive Lattice(ADL), Pseudo-complemented ADL, Stone ADL, Disjunctive ADL, maximal filter, $e$–filter, prime ideal, dense element

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1 Introduction

After Boole’s axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice(ADL) was introduced by Swamy and Rao [10] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an...
ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of $L$ forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. With this motivation, Swamy, Rao and Nanaji[11] introduced the concept of pseudo-complementation on an ADL. They observed that unlike in a distributive lattice, an ADL $L$ can have more than one pseudo-complementation. If $\ast$, $\perp$ are two pseudo-complementations on $L$, it was observed that $x^\ast \lor x^{**}$ is maximal, for all $x \in L$ if and only if $x^\perp \lor x^{+\perp}$ is maximal, for all $x \in L$. With this motivation, in [12], the concept of a Stone ADL was introduced as an ADL with a pseudo-complementation $\ast$ satisfying the condition $x^\ast \lor x^{**}$ is maximal, for all $x \in L$. They studied the properties of pseudo-complemented ADLs and characterized Stone ADLs algebraically, topologically and by means of prime ideals. In [9], Sambasiva Rao introduced $e-$filters in an $MS-$ algebra and proved some related properties. In this paper, $e-$filters are extended to an ADLs with Stone property. Though many results look similar, the proofs are not similar because we do not have the properties like commutativity of $\lor$, commutativity of $\land$ and the right distributivity of $\lor$ over $\land$ in an ADL. We introduced the definition $e-$filter in a Stone ADL in terms of the annihilator ideals. It is observed that set of all dense elements will be an $e-$filter. In addition to this, we obtained the class of all $e-$filters forms a complete distributive lattice. $e-$filter is characterized in element wise. It is established that every maximal filter of Stone ADL is always an $e-$filter. It is also observed that every minimal prime filter which contains a given $e-$filter is an $e-$filter. Finally, we proved some fruitful results on $e-$filters in terms of maximal filters.

2 Preliminaries

First, we recall certain definitions and properties of ADLs, Pseudo-complemented ADLs and Stone ADLs that are required in the paper. We begin with ADL definition as follows.

**Definition 2.1.** [10] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \lor, \land, 0)$ of type $(2, 2, 0)$ satisfying:

1. $(x \lor y) \land z = (x \land z) \lor (y \land z)$
2. $x \land (y \lor z) = (x \land y) \lor (x \land z)$
3. $(x \lor y) \land y = y$
Example 2.2. Every non-empty set \( X \) can be regarded as an ADL as follows. Let \( x_0 \in X \). Define the binary operations \( \vee, \wedge \) on \( X \) by

\[
    x \vee y = \begin{cases} 
        x & \text{if } x \neq x_0 \\
        y & \text{if } x = x_0
    \end{cases}
\]

\[
    x \wedge y = \begin{cases} 
        y & \text{if } x \neq x_0 \\
        x_0 & \text{if } x = x_0
    \end{cases}
\]

Then \( (X, \vee, \wedge, x_0) \) is an ADL (where \( x_0 \) is the zero) and is called a discrete ADL.

If \( (L, \vee, \wedge, 0) \) is an ADL, for any \( a, b \in L \), define \( a \leq b \) if and only if \( a = a \wedge b \) (or equivalently, \( a \vee b = b \)), then \( \leq \) is a partial ordering on \( L \).

Theorem 2.3. [10] If \( (L, \vee, \wedge, 0) \) is an ADL, for any \( a, b, c \in L \), we have the following:

1. \( a \vee b = a \iff a \wedge b = b \)
2. \( a \vee b = b \iff a \wedge b = a \)
3. \( \wedge \) is associative in \( L \)
4. \( a \wedge b \wedge c = b \wedge a \wedge c \)
5. \( (a \vee b) \wedge c = (b \vee a) \wedge c \)
6. \( a \wedge b = 0 \iff b \wedge a = 0 \)
7. \( a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \)
8. \( a \wedge (a \vee b) = a, \quad (a \wedge b) \vee b = b \) and \( a \vee (b \wedge a) = a \)
9. \( a \leq a \vee b \) and \( a \wedge b \leq b \)
10. \( a \wedge a = a \) and \( a \vee a = a \)
11. \( 0 \vee a = a \) and \( a \wedge 0 = 0 \)
12. If \( a \leq c, \quad b \leq c \) then \( a \wedge b = b \wedge a \) and \( a \vee b = b \vee a \)
13. \( a \vee b = (a \vee b) \vee a \).

It can be observed that an ADL \( L \) satisfies almost all the properties of a distributive lattice except the right distributivity of \( \vee \) over \( \wedge \), commutativity of \( \vee \), commutativity of \( \wedge \). Any one of these properties make an ADL \( L \) a distributive lattice. That is
Theorem 2.4. [10] Let \((L, \lor, \land, 0)\) be an ADL with 0. Then the following are equivalent:

1. \((L, \lor, \land, 0)\) is a distributive lattice
2. \(a \lor b = b \lor a\), for all \(a, b \in L\)
3. \(a \land b = b \land a\), for all \(a, b \in L\)
4. \((a \land b) \lor c = (a \lor c) \land (b \lor c)\), for all \(a, b, c \in L\).

As usual, an element \(m \in L\) is called maximal if it is a maximal element in the partially ordered set \((L, \leq)\). That is, for any \(a \in L\), \(m \leq a \Rightarrow m = a\).

Theorem 2.5. [10] Let \(L\) be an ADL and \(m \in L\). Then the following are equivalent:

1. \(m\) is maximal with respect to \(\leq\)
2. \(m \lor a = m\), for all \(a \in L\)
3. \(m \land a = a\), for all \(a \in L\)
4. \(a \lor m\) is maximal, for all \(a \in L\).

As in distributive lattices [1, 3], a non-empty subset \(I\) of an ADL \(L\) is called an ideal of \(L\) if \(a \lor b \in I\) and \(a \land x \in I\) for any \(a, b \in I\) and \(x \in L\). Also, a non-empty subset \(F\) of \(L\) is said to be a filter of \(L\) if \(a \land b \in F\) and \(x \lor a \in F\) for \(a, b \in F\) and \(x \in L\).

The set \(I(L)\) of all ideals of \(L\) is a bounded distributive lattice with least element \(\{0\}\) and greatest element \(L\) under set inclusion in which, for any \(I, J \in I(L), I \cap J\) is the infimum of \(I\) and \(J\) while the supremum is given by \(I \lor J := \{a \lor b \mid a \in I, b \in J\}\). A proper ideal \(P\) of \(L\) is called a prime ideal if, for any \(x, y \in L, x \land y \in P \Rightarrow x \in P\) or \(y \in P\). A proper ideal \(M\) of \(L\) is said to be maximal if it is not properly contained in any proper ideal of \(L\). It can be observed that every maximal ideal of \(L\) is a prime ideal. Every proper ideal of \(L\) is contained in a maximal ideal. For any subset \(S\) of \(L\) the smallest ideal containing \(S\) is given by \((S) := \{\bigvee_{i=1}^{n} s_i \land x \mid s_i \in S, x \in L\ and \ n \in N\}\). If \(S = \{s\}\), we write \((s)\) instead of \((S)\). Similarly, for any \(S \subseteq L, [S) := \{x \lor (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L\ and \ n \in N\}\). If \(S = \{s\}\), we write \([s)\) instead of \([S)\).

Theorem 2.6. [10] For any \(x, y \in L\) the following are equivalent:

1. \((x) \subseteq (y)\)
2. \(y \land x = x\)
For any $x, y \in L$, it can be verified that $(x \vee y) = (x \vee y)$ and $(x \wedge y) = (x \wedge y)$. Hence the set $PI(L)$ of all principal ideals of $L$ is a sublattice of the distributive lattice $I(L)$ of ideals of $L$.

**Theorem 2.7 ([5]).** Let $I$ be an ideal and $F$ a filter of $L$ such that $I \cap F = \emptyset$. Then there exists a prime ideal $P$ such that $I \subseteq P$ and $P \cap F = \emptyset$.

For any $A \subseteq L$, $Ann(A) = \{x \in L \mid a \wedge x = 0 \text{ for all } a \in A\}$ is an ideal of $L$. We write $Ann\{a\}$ for $Ann\{a\}$. Then clearly $Ann\{0\} = L$ and $Ann\{L\} = \{0\}$.

**Definition 2.8.** Let $L$ be an ADL and $x \in L$. Then define $Ann\{x\} = \{y \in L \mid x \wedge y = 0\}$. Clearly, $Ann\{x\}$ is an ideal in $L$ and hence an annihilator ideal.

Annihilators have many important properties. We give some of them in the following lemma which can be proved directly.

**Lemma 2.9.** Let $L$ be an ADL and for any $x, y \in L$. Then we have:

1. $x \leq y \Rightarrow Ann\{y\} \subseteq Ann\{x\}$
2. $Ann\{(x \wedge y)\} = Ann\{(y \wedge x)\}$
3. $Ann\{(x \vee y)\} = Ann\{(y \vee x)\}$
4. $Ann\{(x \vee y)\} = Ann\{x\} \cap Ann\{y\}$
5. $Ann\{x\} \vee Ann\{y\} \subseteq Ann\{(x \wedge y)\}$.

**Definition 2.10 ([11]).** Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \rightarrow a^*$ on $L$ is called a pseudo-complementation on $L$ if, for any $a, b \in L$, it satisfies the following conditions:

1. $a \wedge b = 0 \Rightarrow a^* \wedge b = b$
2. $a \wedge a^* = 0$
3. $(a \vee b)^* = a^* \wedge b^*$

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Here, the unary operation $*$ is called a pseudo-complementation on $L$ and $a^*$ is called a pseudo-complement of $a$ in $L$. An element $a$ of a pseudo-complemented
ADL $L$ is called a dense element if $a^* = 0$. Let us denote the set of all dense elements of $L$ by $D$.

Now we list some results of pseudo-complementation.

**Theorem 2.11** ([11]). Let $L$ be an ADL and $*$, a pseudo-complementation on $L$. Then, for any $a, b \in L$, we have the following:

1. $0^*$ is maximal
2. If $a$ is maximal, then $a^* = 0$
3. $0^{**} = 0$
4. $a^{**} \land a = a$
5. $a^{***} = a^*$
6. $a \leq b \Rightarrow b^* \leq a^*$
7. $a^* \land b^* = b^* \land a^*$
8. $(a \land b)^{**} = a^{**} \land b^{**}$.

**Definition 2.12** ([12]). Let $L$ be an ADL and $*$ a pseudo-complementation on $L$. Then $L$ is called Stone ADL if, for any $x \in L$, $x^* \lor x^{**} = 0^*$.

**Lemma 2.13** ([12]). Let $L$ be a Stone ADL and $a, b \in L$. Then the following conditions hold:

1. $0^* \land a = a$ and $0^* \lor a = 0^*$
2. $(a \land b)^* = a^* \lor b^*$.

### 3 $e-$filters in a Stone ADLs

In [9], M.S Rao introduced the concept of $e-$filters in MS-algebras and studied their properties. In this paper, we extend this concept of $e-$filters to a Stone ADL, analogously and characterized in element wise. Some basic properties of $e-$filters are observed in terms of maximal filters. We proved that every maximal filter of Stone ADL is always an $e-$filter and also observed that every minimal prime filter containing a given $e-$filter is an $e-$filter. Finally, for any filter $F$ of a Stone ADL
L, we derived that $\mathfrak{F}^o(F) = \bigcap \{ \mathfrak{F}(M) \mid M \text{ is a maximal filter and } F \subseteq M \}$. 

Now we begin with the following definition.

**Definition 3.1.** For any filter $F$ of a Stone ADL $L$, define an extension of $F$ as the set $F^e = \{ x \in L \mid x^* \in \text{Ann}\{a\}, \text{ for some } a \in F \}$.

We prove the following result.

**Lemma 3.2.** Let $L$ be a Stone ADL. For any two filters $F$ and $G$ of $L$, we have the following:

1. $F^e$ is a filter of $L$
2. $F \subseteq F^e$
3. $F \subseteq G \Rightarrow F^e \subseteq G^e$
4. $(F \cap G)^e = F^e \cap G^e$
5. $(F^e)^e = F^e$

**Proof.**

1. Let $m$ be any maximal element of $L$. Clearly, $m \in F^e$ and hence $F^e \neq \emptyset$. Let $x, y \in F^e$. Then, $x^* \in \text{Ann}\{a\}$ and $y^* \in \text{Ann}\{b\}$, for some $a, b \in F$. That implies $x^* \land a = 0$ and $y^* \land b = 0$. Now, $(x \land y)^* \land a \land b = (x^* \lor y^*) \land a \land b = (x^* \land a \land b) \lor (y^* \land a \land b) = 0$. Therefore $(x \land y)^* \in \text{Ann}\{a \land b\}$. Since $a \land b \in F$, we get that $x \land y \in F^e$. Let $x \in F^e$ and $r \in L$. Then $x^* \in \text{Ann}\{a\}$, for some $a \in L$. That implies $x^* \land a = 0$. Now $(r \lor x)^* \land a = r^* \lor x^* \land a = 0$. Therefore $(r \lor x)^* \in \text{Ann}\{a\}$ and hence $r \lor x \in F^e$. Thus $F^e$ is a filter of $L$.
2. Let $x \in F$. Clearly, we have $x^* \land x = 0$ and hence $x^* \in \text{Ann}\{x\}$. Therefore $x \in F^e$. Thus $F \subseteq F^e$.
3. Suppose $F \subseteq G$. Let $x \in F^e$. Then, $x^* \in \text{Ann}\{a\}$, for some $a \in F$. Since $F \subseteq G$, we get that $x \in G^e$. Therefore, $F^e \subseteq G^e$.
4. Clearly, $(F \cap G)^e \subseteq F^e \cap G^e$. We prove that $F^e \cap G^e \subseteq (F \cap G)^e$. Let $x \in F^e \cap G^e$. Then, $x^* \in \text{Ann}\{a\}$ and $x^* \in \text{Ann}\{b\}$, for some $a \in F$ and $b \in G$. That implies, $x^* \in \text{Ann}\{a\} \cap \text{Ann}\{b\} = \text{Ann}\{a \lor b\}$. Since $a \lor b \in F \cap G$, we get that $x \in (F \cap G)^e$ and hence $F^e \cap G^e \subseteq (F \cap G)^e$. Therefore $(F \cap G)^e = F^e \cap G^e$.
5. By condition (2), we get that $F^e \subseteq (F^e)^e$. Let $x \in (F^e)^e$. Then $x^* \in \text{Ann}\{a\}$, for some $a \in F^e$. That implies $x^* \land a = 0$ and hence $a^* \land x^* = x^*$. Since $a \in F^e$, we get that $a^* \in \text{Ann}\{b\}$, for some $b \in F$ and hence $a^* \land b = 0$. Now $x^* \land b = a^* \land x^* \land b = 0$. Then $x^* \in \text{Ann}\{b\}$. Therefore $x \in F^e$ and hence $(F^e)^e \subseteq F^e$. Thus $(F^e)^e = F^e$. 

$\square$
Now, we give the definition of $e-$filter in the following.

**Definition 3.3.** A filter $F$ of a Stone ADL $L$ is called an $e-$filter of $L$ if $F = F^e$.

We prove the following result.

**Lemma 3.4.** Let $L$ be a Stone ADL $L$. Then $D$ is the smallest $e-$filter of $L$.

**Proof.** Clearly, $D$ is a filter of $L$ and $D \subseteq D^e$. Let $x \in D^e$. Then $x^* \in Ann\{a\}$, for some $a \in D$. That implies that $x^* \land a = 0$ and hence $a^* \land x^* = x^*$. Therefore $x^* = 0$, since $a$ is a dense element of $L$. That implies $x \in D$ and hence $D = D^e$. Thus $D$ is an $e-$filter of $L$. Suppose $G$ is any $e-$filter of $L$. Let $x \in D$. Then $x^* = 0$. Since $x^* \land 0^* = 0$ and $0^* \in G$, we get that $x \in G = G$. Therefore $D \subseteq G$. Hence $D$ is the smallest $e-$filter of $L$.

From Lemma 3.2, we can observe that the intersection of two $e-$filters of a Stone ADL is again an $e-$filter. But, in general, the supremum of two $e-$filters need not be an $e-$filter. However, in the following, we obtain the class $\mathfrak{F}^e(L)$ of all $e-$filters of $L$ that is a distributive lattice.

**Theorem 3.5.** Let $L$ be a Stone ADL. Then the class $\mathfrak{F}^e(L)$ of all $e-$filters forms a complete distributive lattice on its own.

**Proof.** For any two $e-$filters $F, G$ of $L$, define the ordering $\leq$ on $\mathfrak{F}^e(L)$ such that $F \leq G$ if and only if $F \subseteq G$. Then clearly $(\mathfrak{F}^e(L), \leq)$ is a partially ordered set. Now, consider $F \cap G = (F \cap G)^e$ and $F \cup G = (F \lor G)^e$. Then clearly, $(F \cap G)^e$ is the infimum of both $F^e$ and $G^e$ in $\mathfrak{F}^e(L)$. Clearly, $(F \lor G)^e$ is the upper bound for both $F$ and $G$. Suppose that $H$ is any $e-$filter of $L$ such that $F \subseteq H$ and $G \subseteq H$. Let $x \in (F \lor G)^e$. Then $x^* \in Ann\{a\}$, for some $a \in F \lor G$. Then $x^* \land a = 0$. Since $a \in F \lor G$, we can take $a = f \land g$, for some $f \in F$ and $g \in G$. Since $F \subseteq H$ and $G \subseteq H$, we get that $a \in H$ and hence $x \in H^e$. Since $H$ is an $e-$filter of $L$, we get $x \in H$. Therefore, $(F \lor G)^e$ is the supremum for both $F$ and $G$ in $\mathfrak{F}^e(L)$. Hence, $(\mathfrak{F}^e(L), \cap, \cup, D, L)$ is a bounded lattice. By the extension of the property (4) of lemma 3.2, $(\mathfrak{F}^e(L), \cap, \cup, D, L)$ is a complete lattice. Now for any $F, G, H \in \mathfrak{F}^e(L)$, we obtain $(F^e \cup G^e) \cap (F^e \cup H^e) = (F \lor G)^e \cap (F \lor H)^e = ((F \lor G) \cap (F \lor H))^e = (F \lor (G \cap H))^e = F^e \cup (G \cap H)^e = F^e \cup (G^e \cap H^e)$. Therefore, $(\mathfrak{F}^e(L), \cap, \cup, D, L)$ is bounded and complete distributive lattice.

The following result is a direct consequence of the above.
Corollary 3.6. Every maximal $e-$filter of $\mathfrak{F}(L)$ is a prime $e-$filter.

In the following we characterize $e-$filter element wise.

Theorem 3.7. Let $F$ be a filter of a Stone ADL $L$. Then, the following are equivalent:

1. $F$ is an $e-$filter
2. $x^{**} \in F$ implies $x \in F$, for all $x \in L$
3. For $x, y \in L$, $x^* = y^*$ and $x \in F$ imply that $y \in F$.

Proof. (1) $\Rightarrow$ (2): Assume that $F$ is an $e-$filter of $L$. Let $x$ be any element of $L$ with $x^{**} \in F$. Since $F$ is an $e-$filter of $L$, we get $x^{**} \in F^c$. Then, $(x^{**})^* \in Ann\{a\}$, for some $a \in F$. That implies that $x^* \in Ann\{a\}$ and hence $x \in F^c$. Therefore $x \in F$.

(2) $\Rightarrow$ (3): Assume the condition (2). Let $x, y \in L$ with $x^* = y^*$ and $x \in F$. Then, $x^{**} = y^{**}$. Since $x^{**} = x^{**} \lor x \in F$, we get $y^{**} \in F$. By our assumption we get that $y \in F$.

(3) $\Rightarrow$ (1): Assume the condition (3). We have $F \subseteq F^c$. It is enough to prove that $F^c \subseteq F$. Let $x \in F^c$. Then, $x^* \in Ann\{a\}$, for some $a \in F$. Hence, $x^* \land a = 0$ and hence $(a \lor x)^* = a^* \land x^* = x^*$. Since $a \in F$ and $F$ is a filter of $L$, we get that $a \lor x \in F$. By our assumption we get $x \in F$. Therefore $F^c \subseteq F$. Hence $F = F^c$. Thus $F$ is an $e-$filter of $L$.

Proposition 3.8. Every maximal filter of a Stone ADL is an $e-$filter.

Proof. Let $M$ be a maximal filter of $L$. We prove that $M$ is an $e-$filter of $L$. It is enough to prove that condition (2) of theorem 3.7. Let $x^{**} \in M$. Suppose $x \not\in M$. Then, $M \lor [x] = L$ and hence $0 = a \land x$, for some $a \in M$. That implies $0 = 0^{**} = (a \land x)^{**} = a^{**} \land x^{**}$. Since $M$ is a filter of $L$ and $a \in M$, we get $a^{**} \in M$, and hence $a^{**} \land x^{**} \in M$. Therefore $0 \in M$, Which is a contradiction. Hence $x \in M$. Thus $M$ is an $e-$filter of $L$.

We observed that that every maximal filter of Stone ADL is a prime $e-$filter. Now the following result can be easy to verify.

Lemma 3.9. Let $L$ be a Stone ADL. If $a^* = b^*$ then $(a \lor c)^* = (b \lor c)^*$, $(a \land c)^* = (b \land c)^*$ and $(a)^* = (a \lor b)^*$, for all $a, b, c \in L$.

Theorem 3.10. Let $L$ be a Stone ADL. If $P$ is minimal prime filter of $L$ containing a given $e-$filter, then $P$ is an $e-$filter of $L$. 

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Proof. Let $F$ be an $e$–filter of $L$ and $P$, a minimal in the class of all prime filter of $L$ such that $F \subseteq P$. We have to prove that $P$ is an $e$–filter of $L$. Suppose that $P$ is not an $e$–filter. Then by the theorem 3.7(3), there exists elements $x, y \in L$ such that $x^* = y^*, x \in P$ and $y \not\in P$. Take $I = (L \setminus P) \cup (x \lor y)$. We prove that $I \cap F = \emptyset$. Suppose $I \cap F \neq \emptyset$. Choose $a \in I \cap F$. Since $a \in I$, we get $a = r \lor s$, for some $r \in L \setminus P$ and $s \in (x \lor y)$. Since $s \in (x \lor y)$, we obtain $(x \lor y) \land s = s$. Now, $r \lor s = r \lor [(x \lor y) \land s] = (r \lor x \lor y) \land (r \lor s)$. That implies $a = (r \lor x \lor y) \land a$ and hence $r \lor x \lor y = (r \lor x \lor y) \lor a$. Since $a \in F$, we get that $r \lor x \lor y \in F$. Since $x^* = y^*$, by the above lemma, we obtain $(r \lor x \lor y)^* = r^* \land y^* = r^* \land x^* \land y^* = (r \lor x \lor y)^*$. As $F$ is an $e$–filter and $r \lor x \lor y \in F$, we get $r \lor y \in F$ and hence $r \lor y \in P$. Since $P$ is a prime filter, we get that either $r \in P$ or $y \in P$. Since $r \not\in P$ we get $y \in P$, which is a contradiction to $y \notin P$. Therefore $I \cap F = \emptyset$. So that by Zorn’s lemma, there exists a prime filter $Q$, such that $I \cap Q = \emptyset$ and $F \subseteq Q$; as $I \cap Q = \emptyset$, we obtain $Q \subseteq P$. Since $x \in P$ and $P$ is a filter, we get $x \lor y \in P$ but $x \lor y \not\in Q$. That implies $Q \subset P$. Therefore $P$ is not a minimal in the class of all prime filters containing $F$, which is a contradiction. Therefore, $P$ is an $e$–filter of $L$. □

The following definition is taken from [8].

**Definition 3.11.** An ADL $L$ is said to be a disjunctive ADL if for any $x, y \in L$, $Ann\{x\} = Ann\{y\}$ implies $x = y$.

**Theorem 3.12.** Let $L$ be a Stone ADL. If $L$ is a disjunctive ADL, then every filter of $L$ is an $e$–filter.

Proof. Let $L$ be a disjunctive ADL and $F$ be any filter of $L$. Clearly, we have $F \subseteq F^\circ$. Let $x \in F^\circ$. Then $x^* \in Ann\{a\}$, for some $a \in F$. That implies $x^* \land a = 0$ and hence $a^* \land x^* = x^*$. That implies $(a \lor x)^* = x^{**}$. Since $L$ is a disjunctive ADL, we get $a \lor x = x$. Since $a \in F$, we get $x \in F$. Therefore $F = F^\circ$. Hence $F$ is an $e$–filter of $L$. □

We have the following result.

**Theorem 3.13.** For any prime filter $P$ of a Stone ADL $L$, the set $\mathcal{F}(P) = \{x \in L \mid x^* \notin P\}$ is an $e$–filter of $L$.

Proof. Let $m$ be any maximal element of $L$ and $P$, any prime filter of $L$. Clearly, $m^* \notin P$ and we get that $m \in \mathcal{F}(P)$. Let $x, y \in \mathcal{F}(P)$. Then $x^* \notin P$ and $y^* \notin P$. Since $P$ is a prime filter, we get that $(x \land y)^* = x^* \lor y^* \notin P$. Therefore $x \land y \in \mathcal{F}(P)$. □
Let \( x \in \mathcal{F}(P) \) and \( r \in L \). Then \( x^* \notin P \). We prove that \( r \vee x \in \mathcal{F}(P) \). Suppose \( (r \vee x)^* \in P \). Then \( r^* \wedge x^* \in P \) and hence \( x^* \in P \), which is a contradiction. Therefore \( r \vee x \notin \mathcal{F}(P) \). Thus \( \mathcal{F}(P) \) is a filter of \( L \). Clearly, we have \( \mathcal{F}(P) \subseteq (\mathcal{F}(P))^c \). Conversely, let \( x \in (\mathcal{F}(P))^c \). Then \( x^* \in \text{Ann}\{a\} \), for some \( a \in \mathcal{F}(P) \). That implies \( x^* \wedge a = 0 \) and hence \( a^* \wedge x^* = x^* \). Since \( a \in \mathcal{F}(P) \), we have \( a^* \notin P \) and hence \( a^* \in L \setminus P \). That implies that \( x^* \in L \setminus P \) and hence \( x^* \notin P \). Therefore \( x \notin \mathcal{F}(P) \). Thus \( \mathcal{F}(P) \) is an \( e^-\)filter of \( L \). □

**Corollary 3.14.** For any maximal filter \( M \) of a Stone ADL \( L \), \( \mathcal{F}(M) \) is an \( e^-\)filter of \( L \).

**Theorem 3.15.** For any filter \( F \) of a Stone ADL \( L \), the set \( \mathcal{F}^0(F) = \{ x \in L : [x^*] \vee F = L \} \) is an \( e^-\)filter of \( L \).

**Proof.** Let \( F \) be a filter of \( L \) and \( m \) be any maximal element of \( L \). Now \([m^*] \vee F = [0] \vee F = L \vee F = L \) and hence \( m \in \mathcal{F}(F) \). Therefore \( \mathcal{F}^0(F) \neq \emptyset \). Let \( x, y \in \mathcal{F}^0(F) \). Then \([x^*] \vee F = L = [y^*] \vee F \). Now, \([(x \wedge y^*)] \vee F = \{[(x^*) \cap [y^*]] \} \vee F = \{(x^*) \vee F \} \cap \{[y^*] \vee F \} = L \). Hence \( x \wedge y \in \mathcal{F}^0(F) \). Let \( x \in \mathcal{F}^0(F) \) and \( r \in L \). Then, \([x^*] \vee F = L \). Now, \([(r \vee x^*)] \vee F = [r^* \wedge x^*] \vee F = [r^*] \vee [x^*] \vee F = [r^*] \vee L = L \). Therefore \( r \vee x \in \mathcal{F}^0(F) \) and hence \( \mathcal{F}^0(F) \) is a filter of \( L \). Clearly, we have \( \mathcal{F}^0(F) \subseteq (\mathcal{F}^0(F))^c \). Let \( x \in (\mathcal{F}^0(F))^c \). Then \( x^* \in \text{Ann}\{a\} \), for some \( a \in \mathcal{F}^0(F) \). That implies \( x^* \wedge a = 0 \) and \([a^*] \vee F = L \). Now \([x^*] \vee F = [a^* \wedge x^*] \vee F = [x^*] \vee [a^*] \vee F = [x^*] \vee L = L \). Therefore \( x \in \mathcal{F}^0(F) \) and hence \( \mathcal{F}^0(F) \) is an \( e^-\)filter of \( L \). □

Now, we conclude this paper with the following theorem.

**Theorem 3.16.** Let \( F \) be a filter of a Stone ADL \( L \). Then, we have \( \mathcal{F}^0(F) = \bigcap \{ \mathcal{F}(M) : M \text{ is a maximal filter and } F \subseteq M \} \).

**Proof.** Take \( \mathcal{M} = \bigcap \{ \mathcal{F}(M) : M \text{ is a maximal filter and } F \subseteq M \} \). Let \( M \) be a maximal filter of \( L \) with \( F \subseteq M \) and \( x \in \mathcal{F}^0(F) \). Then, \([x^*] \vee F = L \) and hence \([x^*] \vee M = L \). That implies \( x^* \wedge a = 0 \), for some \( a \in M \) and hence \( x^* \notin M \). Therefore \( x \notin \mathcal{F}(M) \). Hence \( \mathcal{F}^0(F) \subseteq \mathcal{M} \). Conversely, let \( x \in \mathcal{M} \). Then, \( x \in \mathcal{F}(M) \), for all maximal filters \( M \) of \( L \) such that \( F \subseteq M \). Then \( x^* \notin M \). We prove that \([x^*] \vee F = L \). Suppose \([x^*] \vee F \neq L \). Then, there exists a maximal filter \( M_0 \) of \( L \) such that \([x^*] \vee F \subseteq M_0 \). Hence, \( x^* \in M_0 \) and \( F \subseteq M_0 \) which is a contradiction. Thus, \([x^*] \vee F = L \). Therefore, \( x \in \mathcal{F}^0(F) \). Thus \( \mathcal{M} \subseteq \mathcal{F}^0(F) \). Hence \( \mathcal{F}^0(F) = \bigcap \{ \mathcal{F}(M) : M \text{ is a maximal filter and } F \subseteq M \} \). □
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